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# Degree of approximation of function belonging to the Lipschitz class by Euler-Cesaro means of its Fourier series

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ABSTRACT

Bhatt and Kathal (2000) established interesting results on (C,1)(E,1) summability of a Fourier series and its conjugate series. In this paper, the degree of approximation of a function $f \in Lip\alpha$ class by Euler-Cesaro means of its Fourier series has been determined.

## 1. Introduction

A function f∈ Lip α if

$f(x+t)-f(x)=O(|t|^{\alpha})$ for $0<\alpha\leq 1$. (1.1)

The degree of approximation of a function of a function f: R→R by a trigonometric polynomial $t_n$ of order n, denoted by $E_n(f)$, is defined by

$E_n(f)=\|t_n-f\|_{\infty}=\text{Sup}\{\,|t_n(x)-f(x)|\,\}$ (Zygmund[1]) (1.2)

Let f(t) be the $2\pi$- periodic, Lebesgue integrable function of t over the interval $(-\pi,\pi)$ and belonging to Lipα class. The Fourier seriesof f(t) is given by

$$f(t)\sim \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n cosnt+b_n sinnt) \qquad (1.3)$$

If

$$\sigma_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} S_k \to S, \quad \text{as } n\to\infty \qquad (1.4)$$

Then an infinite series $\sum_{n=0}^{\infty}u_n$ with the partial sums $S_n$ is said to be summable Cesaro means of order 1 or (C,1) to the definite number S (Hardy[2])

$$\text{If } E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}S_k \to S, \quad \text{as } n\to\infty \qquad (1.5)$$

Then an infinite series $\sum_{n=0}^{\infty}u_n$ with the partial sums $S_n$ is said to be summability (E,1) to the definite number S. (Knopp[3])

If (E,1) method is superimpose on (C,1) method then, another method of summability (E,1)(C,1) is obtained.

Thus, if

$$t_n^{E_1C_1} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k \to S, \text{ as } n\to\infty \qquad (1.6)$$

then the series $\sum_{n=0}^{\infty}u_n$ is said to be summability by (E,1)(C,1) means or Euler-Cesaro means to S.

We use following notation

$$\Phi_x(t)=f(x+t)+f(x-t)-2f(x) \qquad (1.7)$$

The degree of approximation of functions belonging to Lipα by Cesaro means, Norlund means and Hausdroff means has been discussed by a number of researchers like Alexits[4], Sahney and Goel[5],Chandra[6],Quereshi[7-9] Rhoades[10], Bhatt and Khatal[11] obtain interesting results on (C,1)(E,1) summability of a fourier series and its conjugate series. But till now no work seems to have been done to obtain the degree of approximation of a function f∈Lipα by product summability means of the form Euler- Cesaro means. In an attempt to make a study in this direction, in this paper the degree of approximation of a function $f \in Lip\alpha$ class by Euler – Cesaro means of its Fourier series has been determined in the following form:

## 2. Main Theorem

If f : R→ R is $2\pi$ periodic, Lebesgue integrable on $[-\pi,\pi]$ and Lip$\alpha$ function then its degree of approximation by Euler – Cesaro means of its Fourier series(1.3) satisfies

$$\|t_n^{E_1C_1}(x)-f(x)\|_{\infty} = \begin{cases} O\left(\frac{1}{(n+1)^{\alpha}}\right); & if\ 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right); & if\ \alpha-1, \end{cases}$$

For n = 0,1,2,3........

where $t_n^{E_1C_1}(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k$ i.e. Euler- Cesaro means o Fourier series(1.3).

## 3. Estimate

For the proof of the theorem following estimate is required

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$1-\cos^n\frac{t}{2}\cos(n+2)\frac{t}{2} = O((n+1)^2t^2)$, for $0 \le t \le \frac{1}{n+1}$ (3.1)

For $0 \le t \le \frac{1}{n+1}$

$$1-\cos^n\frac{t}{2}\cos(n+2)\frac{t}{2} = 1-\left\{1-\left(\frac{t^2}{8}+\cdots\right)\right\}^n\left\{1-\frac{(n+2)^2t^2}{8}+\cdots\right\}$$

$$\le 1-\left\{1-\frac{nt^2}{8}\right\}\left\{1-\frac{(n+2)^2t^2}{8}\right\}$$

$$= 1-\left\{1-\frac{nt^2}{8}-\frac{(n+2)^2t^2}{8}+\frac{n(n+2)^2t^2}{64}\right\}$$

$$\le \frac{nt^2}{8}+\frac{(n+2)^2t^2}{8}$$

$$= \frac{t^2}{8} + [n+(n+2)^2]$$

$$\le \frac{t^2}{8}(n+1)(4n+4)$$

$$= \frac{t^2}{2}(n+1)^2$$

$$= O((n+1)^2t^2)$$

**4. Proof of the theorem**

Following Titchmarsh [12] the $n^{th}$ partial sum $S_n(x)$ of the series (1.3) at t=x is

$$S_n(x) = f(x) + \frac{1}{2}\int_0^\pi \emptyset(t)\frac{\sin\left(n+\frac{1}{2}\right)}{\sin\frac{1}{2}}dt$$

So the (C,1) means of the series (1.3) are

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(x) \qquad (n=0,1,2,3,........)$$

$$= f(x) + \frac{1}{2\pi(n+1)}\int_0^\pi \frac{\emptyset(t)}{\sin\frac{t}{2}}\left(\sum_{k=0}^{n}\sin\left(k+\frac{1}{2}\right)t\right)dt$$

$$= f(x) + \frac{1}{2\pi(n+1)}\int_0^\pi \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\{f(x+t)+f(x-t)-2f(x)\}dt$$

$$= f(x) + \frac{1}{2\pi(n+1)}\int_0^\pi \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\emptyset(t)dt$$

Now (E,1) transform of (C,1), denoted by $t_n^{E_1C_1}(x)$, is given by

$$t_n^{E_1C_1}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k(x)$$

$$= f(x) + \frac{1}{2^{n+1}\pi(n+1)}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{\sum_{k=0}^{n}\binom{n}{k}\sin^2(k+1)t\right\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{\sum_{k=0}^{n}\binom{n}{k}(1-\cos(k+1)t)\right\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{\sum_{k=0}^{n}\binom{n}{k}-\sum_{k=0}^{n}\binom{n}{k}\cos(k+1)t\right\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\{2^n - Re(e^{it}(1+e^{it})^n)\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\{2^n - Re(e^{it}(1+\cos t+i\sin t)^n)\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{2^n - Re\left(e^{it}\left(2\cos^2\frac{t}{2}+2i\sin\frac{t}{2}\cos\frac{t}{2}\right)^n\right)\right\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{2^n - 2^n\cos^n\frac{t}{2}Re\left(e^{it}\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)^n\right)\right\}dt$$

$$= f(x) + \frac{1}{2^{n+2}(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}2^n\left\{1-\cos^n\frac{t}{2}\left(\cos t\cos\frac{nt}{2}-\sin t\sin\frac{nt}{2}\right)\right\}dt$$

$$= f(x) + \frac{1}{4(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{1-\cos^n\frac{t}{2}\cos\left(t+\frac{nt}{2}\right)\right\}dt$$

$$= f(x) + \frac{1}{4(n+1)\pi}\int_0^\pi \frac{\emptyset(t)}{\sin^2\frac{t}{2}}\left\{1-\cos^n\frac{t}{2}\cos(n+2)\frac{t}{2}\right\}dt$$

Since here $\sin\frac{t}{2} \ge \frac{t}{2}$ it follows that

$$\left|t_n^{E_1C_1}(x)-f(x)\right| \le \frac{\pi}{4(n+1)}\int_0^\pi\left|\frac{\emptyset(t)}{t^2}\right|\left(1-\cos^n\frac{t}{2}\cos(n+2)\frac{t}{2}\right)dt$$

$$= \frac{\pi}{4(n+1)}\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\left|\frac{\emptyset(t)}{t^2}\right|\left(1-\cos^n\frac{t}{2}\cos(n+2)\frac{t}{2}\right)dt$$

$= I_1+I_2$ , say (4.1)

Applying fact that $f \in Lip\alpha$ and estimate (3.1), we have

$$|I_1| \le \frac{\pi}{4(n+1)}\int_0^{\frac{1}{n+1}}\frac{O(t^\alpha)}{t^2}O((n+1)^2t^2)\,dt$$

$$= O(n+1)\int_0^{\frac{1}{n+1}} t^\alpha\,dt$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right); \quad 0<\alpha<1 \qquad (4.2)$$

Let us consider $I_2$

$$|I_2| \le \frac{\pi}{4(n+1)}\int_{\frac{1}{n+1}}^{\pi}\left|\frac{\emptyset(t)}{t^2}\right|\left(1-\cos^n\frac{t}{2}\cos(n+2)\frac{t}{2}\right)dt$$

$$= O\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi}\frac{O(t^\alpha)}{t^2}O(1)dt$$

$$= O\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2}dt$$

$$= O\left(\frac{1}{n+1}\right)\begin{cases}\left(\frac{t^{\alpha-1}}{\alpha-1}\right)_{\frac{1}{n+1}}^{\pi}; & \alpha\ne1\\ (\log t)_{\frac{1}{n+1}}^{\pi}; & \alpha=1\end{cases}$$

$$I_2 = O\left(\frac{1}{n+1}\right)\begin{cases}\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{(n+1)^{\alpha-1}}-\frac{1}{\pi^{1-\alpha}}\right); & \alpha\ne1\\ \log\pi+\log(n+1); & \alpha=1\end{cases}$$

$$= \begin{cases}O\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{(n+1)^\alpha}-\frac{1}{\pi^{1-\alpha}(n+1)}\right); & \alpha\ne1\\ O\left(\frac{\log(n+1)\pi}{n+1}\right); & \alpha=1\end{cases} \qquad (4.3)$$

Combining (4.1),(4.2) and (4.3), we have

$$\left|t_n^{E_1C_1}(x)-f(x)\right| = \begin{cases}O\left(\frac{1}{(n+1)^\alpha}\right); & \alpha\ne1\\ O\left(\frac{\log(n+1)\pi e}{n+1}\right); & \alpha=1\end{cases} \qquad (4.4)$$

Thus

$$\left\|t_n^{E_1C_1}(x)-f(x)\right\|_\infty = \sup_{-\pi\le x\le\pi}\left|t_n^{E_1C_1}(x)-f(x)\right|$$

$$= \begin{cases}O\left(\frac{1}{(n+1)^\alpha}\right); & \alpha\ne1\\ O\left(\frac{\log(n+1)\pi e}{n+1}\right); & \alpha=1\end{cases}$$

**5. Example:** Consider the infinite series

$1 + 2\sum_{n=1}^{\infty} n(-1)^{n+1}$ (5.1)

For this series ,

$S_0 = 1$, $S_1 = 3$, $S_2 = -1$, $S_3 = 5$, $S_4 = -3$, ................

Thus we get that

$$S_n = \begin{cases} (1-n) & ;when\ n\ is\ even \\ (n+2) & ;when\ n\ is\ odd \end{cases}$$

It is known that

$\sigma_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} S_k,$

*Then*

$\sigma_n^1 = 1, \quad \sigma_n^1 = 2, \quad \sigma_n^1 = 1, \quad \sigma_n^1 = 2, \ldots\ldots\ldots\ldots$

$$i.e. \quad \sigma_n^1 = \begin{cases} 1\ ;when\ n\ is\ even \\ 2\ ;when\ n\ is\ odd \end{cases}$$

Hence $\lim_{n\to\infty}\sigma_n^1$ does not exists. Therefore the series (5.1) is not (C,1) summabile

Let is consider (E,1) (C,1) summability of series (5.1)

$$t_n^{E_1C_1}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k(x)$$

$$=\frac{1}{2^n}\left\{\binom{n}{0}\sigma_0 + \binom{n}{1}\sigma_1 + \cdots.\binom{n}{n}\sigma_n\right\} = \frac{3}{2}$$

Then

$\lim_{n\to\infty} \quad t_n^{E_1C_1} = \frac{3}{2}$

Thus the series (5.1) is (E,1)(C,1) summable to the sum $\frac{3}{2}$

Therefore the product summability (E,1)(C,1) is more powerful than the individual method. Consequently (E,1)(C,1) means gives better approximation than individual method.

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